The WKB method and geometric instability for non linear Schrödinger equations on surfaces

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1 Introduction

Let (M, g) be a Riemannian surface (i.e. a Riemannian manifold of dimension 2), orientable or not. We assume that M is either compact or a compact perturbation of the euclidian space, so that the Sobolev embeddings are true. Consider $\Delta = \Delta_g$ the Laplace-Beltrami operator. In this paper we are interested in constructing WKB approximations for the non linear cubic Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = \varepsilon |u|^2 u(t,x), \quad \varepsilon = \pm 1 \\ u(0,x) = u_0(x) \in H^\sigma(M) \end{cases} \tag{1.1}$$

that is, given a small parameter $0 < h < 1$ and an integer N, functions $u_N(h)$ satisfying

$$i\partial_t u_N(h) + \Delta u_N(h) = \varepsilon |u_N(h)|^2 u_N(h) + R_N(h) \tag{1.2}$$

with $\|u_N(h)\|_{H^\sigma} \sim 1$ and $\|R_N(h)\|_{H^\sigma} \leq C_N h^N$.
Here h is introduced so that $u_N(h)$ oscillates with frequency $\sim \frac{1}{h}$.
These approximate solutions to (1.1) will lead to some instability properties in the following sense (where h^{-1} will play the role of n):

Definition 1.1 *We say that the Cauchy problem (1.1) is unstable near 0 in $H^\sigma(M)$, if for all $C > 0$ there exist times $t_n \longrightarrow 0$ and $u_{1,n}, u_{2,n} \in H^\sigma(M)$ solutions of (1.1) so that*

$$\begin{aligned} \|u_{1,n}(0)\|_{H^\sigma(M)},\ \|u_{2,n}(0)\|_{H^\sigma(M)} &\leq C \\ \|u_{1,n}(0) - u_{2,n}(0)\|_{H^\sigma(M)} &\longrightarrow 0 \\ \limsup \|u_{1,n}(t_n) - u_{2,n}(t_n)\|_{H^\sigma(M)} &\geq \frac{1}{2}C \end{aligned}$$

when $n \longrightarrow +\infty$.

This means that the problem is not uniformly well posed, if we refer to the following definition:

Definition 1.2 *Let $\sigma \in \mathbb{R}$. Denote by $B_{R,\sigma}$ the ball of radius R in H^σ. We say that the Cauchy problem (1.1) if uniformly well posed in H^σ if the flow map*

$$u_0 \in B_{R,\sigma} \cap H^1(M) \longmapsto \Phi_t(u_0) \in H^\sigma(M)$$

is uniformly continuous for any t.

We now state our instability result:

Proposition 1.3 *Let $0 < \sigma < \frac{1}{4}$, and assume that M has a stable and non degenerated periodic geodesic (see Assumptions 1.4 and 1.5), then the Cauchy problem (1.1) is not uniformly well-posed.*

This problem is motivated by the following results: In [3], N. Burq, P. Gérard and N. Tzvetkov show that (1.1) is unstable on the sphere $\mathbb{S}^2$ for $0 < \sigma < \frac{1}{4}$. In fact they construct solutions of (1.1) of the form

$$u_n^{\kappa}(t,x) = \kappa \mathrm{e}^{i\lambda_n^{\kappa} t}(n^{\frac{1}{4}-\sigma}\psi_n(x) + r_n(t,x)) \tag{1.3}$$

where $0 < \kappa < 1$, $\psi_n = (x_1 + ix_2)^n$ is a spherical harmonic which concentrates on the equator of the sphere when $n \longrightarrow +\infty$ and where r_n is an error term which is small. To obtain instability, they consider $\kappa_n \longrightarrow \kappa$, then

$$\|u_{1,n}(0) - u_{2,n}(0)\|_{H^{\sigma}(\mathbb{S}^2)} \lesssim |\kappa' - \kappa| \longrightarrow 0$$

but

$$\|u_n^{\kappa}(t_n) - u_n^{\kappa'}(t_n)\|_{H^{\sigma}(\mathbb{S}^2)} \gtrsim \kappa|\mathrm{e}^{i\lambda_n^{\kappa}t_n} - \mathrm{e}^{i\lambda_n^{\kappa'}t_n}| \longrightarrow 2\kappa$$

with a suitable choice of $t_n \longrightarrow 0$.
We follow this strategy but as the surface is not rotation invariant, the ansatz will be more complicated than (1.3).
This result is sharp, because in [4] they show that (1.1) is uniformly well posed on $\mathbb{S}^2$ when $\sigma > \frac{1}{4}$.
On the other hand, in [2] J. Bourgain shows that (1.1) is uniformly well posed on the rational thorus $\mathbb{T}^2$ when $\sigma > 0$.
These results show how the geometry of M can lead to instability fot the equation (1.1). Therefore it seems reasonable to obtain a result like Proposition 1.3 with purely geometric assumptions.

We first make the following assumption on M:

Assumption 1.4 *The manifold M has a periodic geodesic.*

Denote by γ such a geodesic, then there exists a system of coordinates (r,s) near γ, say for $(r,s) \in]-r_0, r_0[\times\mathbb{S}^1$, called Fermi coordinates such that (see [9], p. 80)

(i) The curve $r = 0$ is the geodesic γ parametrized by arclength and

(ii) The curves $s =$ constant are geodesics parametrized by arclength. The curves $r =$ constant meet these curves orthogonally .

(iii) In this system the metric writes

$$g = \begin{pmatrix} 1 & 0 \\ 0 & a^2(s,r) \end{pmatrix}.$$

We set the length of γ equal to 2π. Denote by $R(r,s)$ the Gauss curvature at (r,s), then a is the unique solution of

$$\begin{cases} \frac{\partial^2 a}{\partial r^2} + R(r,s)a = 0 \\ a(0,s) = 1, \frac{\partial a}{\partial r}(0,s) = 0. \end{cases} \tag{1.4}$$

The initial conditions traduce the fact that the curve $r = 0$ is a unit-speed geodesic. In these coordinates the Laplace-Beltrami operator is

$$\Delta := \frac{1}{\sqrt{\det g}} \mathrm{div}(\sqrt{\det g}\; g^{-1}\nabla) = \frac{1}{a}\partial_s(\frac{1}{a}\partial_s) + \frac{1}{a}\partial_r(a\partial_r).$$

A function on M, defined locally near γ, can be identified with a function of $[0,2\pi]\times]-r_0, r_0[$ such that

$$\forall (s,r) \in [0,2\pi]\times]-r_0,r_0[\quad f(s+2\pi, r) = f(s, \omega r)$$

where $\omega = 1$ if M is orientable and $\omega = -1$ if M is not. Define

$$\omega_1 = \frac{1}{2}(\omega - 1) \in \{-1, 0\}. \tag{1.5}$$

From (1.4) we deduce that a admits the Taylor expansion

$$a = 1 - \frac{1}{2}R(s)r^2 + R_3(s)r^3 + \cdots + R_p(s)r^p + o(r^p), \tag{1.6}$$

and as $a(s+2\pi, r) = a(s, \omega r)$, we deduce $R(s+2\pi) = R(s)$ and for all $j \geq 3$, $R_j(s+2\pi) = \omega^j R_j(s)$.
Let $p_2 = \frac{1}{a^2}\sigma^2 + \rho^2$ be the principal symbol of Δ, and

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}s(t) = \frac{\partial p_2}{\partial \sigma} = \frac{2\sigma}{a^2}, & \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = -\frac{\partial p_2}{\partial s} = -\partial_s(\frac{1}{a^2})\sigma^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}r(t) = \frac{\partial p_2}{\partial \rho} = 2\rho, & \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\frac{\partial p_2}{\partial r} = -\partial_r(\frac{1}{a^2})\sigma^2 \\ s(0) = s_0,\ \sigma(0) = \sigma_0, & r(0) = r_0,\ \rho(0) = \rho_0, \end{cases} \tag{1.7}$$

its associated hamiltonian system, where $p_2 = p_2(s(t), r(t), \sigma(t), \rho(t))$. The system (1.7) admits a unique solution and defines the hamiltonian fow

$$\Phi_t : (s_0, \sigma_0, r_0, \rho_0) \longmapsto (s(t), \sigma(t), r(t), \rho(t)).$$

The curve $\Gamma = \{(s(t) = t, \sigma(t) = 1/2, r(t) = 0, \rho(t) = 0), t \in [0, 2\pi]\}$ is solution of (1.7) and its projection in the (r,s) space is the curve γ. Now denote by ϕ the Poincaré map associated to the trajectory Γ and to the hyperplane $\Sigma = \{s = 0\}$. There exists a neighborhood $\mathcal{N}$ of $(\sigma = 1/2, r = 0, \rho = 0)$ such that the following makes sense: solve the system (1.7) with the initial conditions $(0, \sigma_0, r_0, \rho_0) \in \{0\} \times \mathcal{N}$ and let T be such that $s(T) = 2\pi$, then ϕ is the application

$$\phi : (r_0, \rho_0) \longmapsto (r(T), \rho(T)).$$

Moreover, the Poincaré map is continuously differentiable (see [10] p. 193). To obtain its differential $\mathrm{d}\phi(0,0)$ at $(0,0)$, we linearize the system (1.7) about the orbit Γ, i.e.

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}s(t) = 2\sigma, & \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}r(t) = 2\rho, & \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\frac{1}{2}R(s(t))r, \end{cases} \tag{1.8}$$

then $\sigma = \frac{1}{2}$, $s(t) = t$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} r \\ \rho \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -R/2 & 0 \end{pmatrix}\begin{pmatrix} r \\ \rho \end{pmatrix}. \tag{1.9}$$

Hence the application $\mathrm{d}\phi(0,0)$ is

$$\mathrm{d}\phi(0,0) : (r_0, \rho_0) \longmapsto (r(2\pi), \rho(2\pi)),$$

where (r, ρ) solve (1.9). As $\mathrm{d}\phi(0,0)$ is symplectic, it admits two eigenvalues μ and μ^{-1} that are called the characteristic multipliers of the system (1.9). We add the following assumption on γ, which can be formulated in terms of the eigenvalues of $\mathrm{d}\phi(0,0)$:

Assumption 1.5 *The geodesic γ is stable, i.e. $d\phi(0,0)$ is a rotation. Then the multipliers take the form $\mu = e^{i\lambda}$ and $\mu^{-1} = e^{-i\lambda}$ with $\lambda \in \mathbb{R}$. We assume moreover that there exist $\tau, \mu > 0$ such that*

$$\forall (p,q) \in \mathbb{Z} \times \mathbb{N} \quad |p - q\frac{\lambda}{\pi}| \geq \frac{\mu}{|(p,q)|^{\tau}}, \tag{1.10}$$

where $|(p,q)| = |p| + |q|$. When this condition is fulfilled, we say that γ is non degenerated.

Remark 1.6 *Almost every $\lambda \in \mathbb{R}$ satisfies* (1.10) *with $\tau > 1$. This is an easy consequence of [1] p. 159, e.g.*

Notice that the function r which satisfies (1.9) is solution of

$$\ddot{y}(s) + R(s)y(s) = 0. \tag{1.11}$$

Consider a_0 the solution of 1.11 with initial conditions $a_0(0) = 1$ and $\dot{a}_0(0) = i$. Then, from the Floquet theory, there exists a 2π-periodic function P so that

$$a_0(s) = \mathrm{e}^{i\frac{\lambda}{2\pi}s} P(s)$$

(or $a_0(s) = \exp(-i\frac{\lambda}{2\pi}s)P(s)$, but λ can be replaced with $-\lambda$).
Here, and in all the paper we denote by $\dot{f} = \frac{\mathrm{d}}{\mathrm{d}s}f$ if f is differentiable. This notation is motived by the fact that s will play the role of a time variable (see section 2).

In order to prove Proposition 1.3, we construct stationnary approximate solutions of (1.1), as stated in the following Theorem

Theorem 1.7 *Assume 1.4 and 1.5. Let $h \in]0,1]$ such that $\frac{1}{h} \in \mathbb{N}$, let $\kappa, \sigma > 0$ and $k \in \mathbb{N}$. Let λ be given by Assumption 1.5 and ω_1 by* (1.5).
Define $E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$ and $\delta = \kappa h^{\sigma}$.
Then for all $N \in \mathbb{N}$, there exist $\lambda_N(k) \in \mathbb{R}$ and a family $u_N(h)$ such that $C_1\delta \leq \|u_N(h)\|_{L^2(M)} \leq C_2\delta$ with $C_1, C_2 > 0$ independent of N and h, and

$$-\Delta u_N(h) = \lambda_N(k)u_N(h) - \varepsilon|u_N(h)|^2 u_N(h) + h^N g_N(h) \tag{1.12}$$

with for all $N \in \mathbb{N}$
$$\|h^N g_N(h)\|_{H^n(M)} \lesssim h^{N-n}.$$
Moreover
$$\lambda_N(k) = \frac{1}{h^2} - \frac{2}{h}E_0(k) + \frac{1}{\sqrt{h}}\varepsilon\delta^2 C_0 + \mathcal{O}(1),$$
where $C_0 > 0$ *is independent of* ε *and* δ.

REMARK 1.8 *The analog of Theorem 1.7 was proved by J. Ralston in* [11] *for the linear case* ($\varepsilon = 0$)*, with the same type of assumptions.*

REMARK 1.9 *Consider the more general equations*

$$i\partial_t u + \Delta u = F(u), \tag{1.13}$$

where $F : \mathbb{C} \longrightarrow \mathbb{C}$ *is a* $\mathcal{C}^\infty$ *function. The result of Theorem 1.7 is likely to hold with other nonlinearities* $F(u)$*, for example for* $F(z) = z^3$*,* $F(z) = z^4$ *or* $F(z) = (1+|z|^2)^\alpha z$ *with* $\alpha < 1$*. However, the instability phenomenon is strongly related to the gauge invariance of the equation* (1.13)*.*

REMARK 1.10 *The restriction* (1.10) *seems to be purely technical. For instance, when* $M = \mathbb{S}^2$*,* $\lambda = 0$ *but the result holds (see [3]).*

The scheme of the paper is the following: Thanks to a scalling, we reduce the problem (1.12) to the resolution of linear Schrödinger equations with a harmonic time dependent potential, and we will see, using Assumption 1.5, that these equations have periodic solutions. To prove Proposition 1.3 we show that the family $u_N(h)$ provide good approximations of (1.1) in times where instability occurs.

NOTATIONS 1.11 *In this paper* c*,* C *denote constants the value of which may change from line to line. We use the notations* $a \sim b$*,* $a \lesssim b$ *if* $\frac{1}{C}b \leq a \leq Cb$*,* $a \leq Cb$ *respectively. By* $\delta_{i,j}$ *we mean the Kronecker symbol, i.e.* $\delta_{i,j} = 0$ *for* $i \neq j$ *and* $\delta_{i,i} = 1$*.*

REMARK 1.12 *In the sequel we do not always mention the dependence on* h *of the functions: we will write* u*,* f*,* r_i*,* $\ldots$ *instead of* u_h*,* f_h*,* $r_{i,h}$*,* $\ldots$

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2 The WKB construction

Consider the equation

$$-\Delta u = \lambda u - \varepsilon |u|^2 u. \tag{2.1}$$

Given $h > 0$, we are looking for a solution of the form

$$u = \delta h^{-\frac{1}{4}} \mathrm{e}^{i\frac{s}{h}} f(s, r, h) \tag{2.2}$$

where $\delta = \kappa h^{\sigma}$, with $\kappa > 0$ and $0 \le \sigma \le \frac{1}{4}$. We try to find a solution of (2.1) of the form $u \sim \sum_{j\ge 0} h^{j/2} u_j$. Thus we write $\lambda \sim h^{-2} \sum_{j\ge 0} h^{j/2}\lambda_j$. As we will see, identifying each power of h will lead to a linear equation which can be solved with a suitable choice of λ_j.

Choose h such that $h^{-1} \in \mathbb{N}$, this ensures that $\exp i\frac{s}{h}$ is $2\pi-$periodic. Such a condition on h is natural and is known as a Bohr-Sommerfeld quantification condition.

With the ansatz (2.2), equation (2.1) becomes

$$\begin{aligned} -\frac{1}{a^2}\Big(\frac{2i}{h}\partial_s f + \partial_s^2 f - \frac{1}{h^2} f\Big) - \frac{1}{a}\partial_s\Big(\frac{1}{a}\Big)\Big(\frac{i}{h} f + \partial_s f\Big) \\ -\partial_r^2 f - \frac{\partial_r a}{a}\partial_r f = \lambda f - \varepsilon\delta^2 h^{-\frac{1}{2}} |f|^2 f. \end{aligned} \tag{2.3}$$

We make the change of variables $x = \frac{r}{\sqrt{h}}$ and set $v(s, x, h) = f(s, \sqrt{h}x, h)$. Thus $\partial_r f = \frac{1}{\sqrt{h}}\partial_x v$ and $\partial_r^2 f = \frac{1}{h}\partial_x^2 v$. Therefore we now have to find $v \sim \sum_{j\ge 0} h^{j/2} v_j$. Using (1.6) we obtain the following Taylor expansions in h

$$\frac{1}{a^2} = 1 + hRx^2 - 2h^{\frac{3}{2}} R_3 x^3 + \mathcal{O}(h^2),$$

$$a^{-1}\partial_s(a^{-1}) = \mathcal{O}(h) \quad \text{and} \quad a^{-1}\partial_r a = \mathcal{O}(h^{\frac{1}{2}}).$$

Equation (2.3) can therefore be written, after multiplication by $\frac{1}{2}h$

$$i\partial_s v + \frac{1}{2}\partial_x^2 v - \frac{1}{2}Rx^2 v = \frac{1-\lambda h^2}{2h} v + h^{\frac{1}{2}} R_3 x^3 v + \frac{1}{2}\varepsilon\delta^2 h^{\frac{1}{2}} |v|^2 v + hPv \tag{2.4}$$

where

$$P = A_1\partial_s^2 + A_2\partial_s + A_3\partial_r + A_4 \tag{2.5}$$

is a second order differential operator with coefficients $A_j = A_j(s, x, h)$ satisfying $A_j(s + 2\pi, x, h) = A_j(s, \omega x, h)$ for $0 \le j \le 4$.

Denote by $E = \frac{1-\lambda h^2}{2h} = E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p + o(h^{\frac{p}{2}})$ and write $v = v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p + o(h^{\frac{p}{2}})$ and by identifying the powers of h we obtain the system of equations:

$$\begin{aligned}
\Big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\Big) v_0 &= 0 && (2.6)\\
\Big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\Big) v_1 &= E_1 v_0 + R_3 x^3 v_0 + \frac{1}{2}\varepsilon\delta^2 |v_0|^2 v_0 && (2.7)\\
\cdots &= \cdots \\
\Big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2 - E_0\Big) v_p &= +E_p v_0 + Q_p. && (2.8)\\
\cdots &= \cdots
\end{aligned}$$

so that the $(j+1)$th equation of unknown (v_j, E_j) correspond to the annihilation of the coefficient of $h^{\frac{j}{2}}$ in (2.4).
Here Q_p is a function which only depends on x, s, $(v_j)_{j\leq p-1}$ and $(E_j)_{j\leq p-1}$.

REMARK 2.1 *Notice that thanks to the scalling, we have reduced the problem (2.1) to the resolution of linear equations. However we have to solve them exactly; no smallness assumption on x is possible, as x can be of size $\sim \frac{1}{\sqrt{h}}$.*

Once we have solved the previous equations, we will be able to construct approximate solutions of (2.1); more precisely, we will obtain the following proposition, which is the main result of this section.

PROPOSITION 2.2 *Let $\chi \in \mathcal{C}_0^\infty(]-r_0, r_0[)$ be such that $0 \leq \chi \leq 1$, $\chi = 1$ on $[-r_0/2, r_0/2]$ and suppose moreover that χ is an even function. Denote by*

$$u_p = \delta h^{-\frac{1}{4}} \chi(r) e^{i\frac{s}{h}} (v_0 + h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p}{2}} v_p)(s, \frac{r}{\sqrt{h}}) \tag{2.9}$$

and by

$$\lambda_p = \frac{1}{h^2} - \frac{2}{h}(E_0 + h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p}{2}} E_p). \tag{2.10}$$

Then u_p satisfies $\|u_p\|_{L^2(M)} \sim \delta$ and

$$-\Delta u_p = \lambda_p u_p - \varepsilon |u_p|^2 u_p + h^{\frac{p-1}{2}} g_p(h) \tag{2.11}$$

with

$$\forall h \in]0,1], \; \forall n \in \mathbb{N}, \quad \|h^{\frac{p-1}{2}} g_p(h)\|_{H^n} \lesssim \delta h^{\frac{p-1}{2} - n}.$$

2.1 Preliminaries: the analysis of the linear equations

We will solve the system (2.6) for $x \in \mathbb{R}$, even if the function a is only defined for $|r| \leq r_0$ i.e. for $x \leq \frac{r_0}{\sqrt{h}}$, but the equations (2.6) make sense for $x \in \mathbb{R}$. At the end we will use a cutoff argument to construct a proper function on M.
Consider the Hilbertian basis of $L^2(\mathbb{R})$ composed of the Hermite functions $(\varphi_k)_{k\geq 0}$ which are the eigenfunctions of the harmonic oscillator $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$, i.e. $H\varphi_k = (k + \frac{1}{2})\varphi_k$. Moreover $\varphi_k(x) = P_k(x)\mathrm{e}^{-x^2/2}$ where P_k is a polynomial of degree k with $P_k(-x) = (-1)^k P_k(x)$. The link between the s-dependent operator $-\frac{1}{2}\partial_x^2 + \frac{1}{2}R(s)x^2$ and H is given by the following result proved by M. Combescure in [7].

THEOREM 2.3 *Let $a_0 : \mathbb{R} \longrightarrow \mathbb{C}$ be the solution of (1.11) with $a_0(0) = 1$, $\dot{a}_0(0) = i$. Define*

$$\alpha = \log |a_0|, \; \beta = \frac{1}{2i} \log \frac{a_0}{\overline{a_0}},$$

let the unitary transform $T(s)$ be defined by

$$T(s) = e^{i\dot{\alpha}(s)x^2/2} e^{-i\alpha(s)D}, \; \textit{where } D = -\frac{i}{2}(x \cdot \nabla + \nabla \cdot x),$$

and let $U(s,\tau)$ be the unitary evolution operator for $-\frac{1}{2}\partial_x^2 + \frac{1}{2}R(s)x^2$, i. e. $U(s,\tau)\varphi$ is the unique solution of the problem

$$\left\{ \begin{array}{l} (i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}R(s)x^2)u = 0 \\ u(\tau,x) = \varphi(x) \in L^2(\mathbb{R}) \end{array} \right.$$

Then we have for any $s,\tau \in \mathbb{R}$

$$U(s,\tau) = T(s)e^{-i(\beta(s)-(\beta(\tau))H}T(\tau)^{-1}.$$

REMARK 2.4 *The functions α and β are well defined: suppose that there exists s_0 such that $a_0(s_0) = 0$, then $Re(a_0)$ and $Im(a_0)$ are linearly dependent, which is impossible with this choice of the initial conditions.*

REMARK 2.5 *Define $\theta(s) = \beta(s) - \frac{\lambda}{2\pi}s$ where λ is given by Assumption 1.5. Then α and θ are 2π-periodic real functions. Moreover $\alpha(0) = \dot{\alpha}(0) = \beta(0) = \theta(0) = 0$.*

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space, i.e. the space of smooth functions which are fast decreasing and their derivatives too.

PROPOSITION 2.6 *Let $\psi_0 \in \mathcal{S}(\mathbb{R})$ and $E \in \mathbb{C}$. Let $f \in \mathcal{C}^\infty([0,2\pi]\times\mathbb{R},\mathbb{R})$ be such that*

$$\forall n \in \mathbb{N},\ \forall s \in [0,2\pi], \quad \partial_s^n f(s,\cdot) \in \mathcal{S}(\mathbb{R}),$$

in other words $f \in \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$.
Let $\psi \in \mathcal{C}^1\left([0,2\pi],L^2(\mathbb{R})\right) \cap \mathcal{C}^0\left([0,2\pi],H^2(\mathbb{R})\right)$ be the solution of

$$\left\{ \begin{array}{l} i\partial_s\psi + \frac{1}{2}\partial_x^2\psi - \frac{1}{2}Rx^2\psi - E\psi = f \\ \psi(0,x) = \psi_0(x). \end{array} \right. \tag{2.12}$$

Then $\psi \in \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$.

Proof: By replacing ψ with $\mathrm{e}^{iEt}\psi$, we can assume that $E=0$. The solution of equation (2.12) is given by

$$\begin{aligned} \psi(s,\cdot) &= U(s,0)\psi_0 - i\int_0^s U(s,\tau)f(\tau,\cdot)\mathrm{d}\tau \\ &= T(s)\mathrm{e}^{-i\beta(s)H}\left(\psi_0 - i\int_0^s \mathrm{e}^{i\beta(\tau)H}T(\tau)^{-1}f(\tau,\cdot)\mathrm{d}\tau\right). \end{aligned} \tag{2.13}$$

As D is a transport operator, we have

$$T, T^{-1} : \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R})) \longrightarrow \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R})),$$

we only have to show that

$$\mathrm{e}^{i\beta H} : \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R})) \longrightarrow \mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R})).$$

This follows from the fact that β is regular and $\mathrm{e}^{iH} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$.

The description of U given in Theorem 2.3 yields the following representation of $U(s,0)\varphi_k$:

Proposition 2.7 *For all $k \in \mathbb{N}$ and $s, x \in \mathbb{R}$ we have*

$$U(s,0)\varphi_k(x) = e^{i\dot{\alpha}(s)x^2/2} e^{-i(\frac{1}{2}+k)\beta(s)} e^{-\frac{1}{2}\alpha(s)} \varphi_k\left(xe^{-\alpha(s)}\right). \tag{2.14}$$

Proof: According to Theorem 2.3, and as $H\varphi_k = (k+\frac{1}{2})\varphi_k$,

$$U(s,0)\varphi_k = \mathrm{e}^{i\dot{\alpha}(s)x^2/2}\mathrm{e}^{-i(k+\frac{1}{2})\beta(s)}\mathrm{e}^{-i\alpha(s)D}\varphi_k.$$

Denote by $f(s) = \mathrm{e}^{-i\alpha(s)D}\varphi_k$. Then f is solution of the transport equation

$$\partial_s f = -\frac{1}{2}\dot{\alpha}(s)\big(x\partial_x f + \partial_x(xf)\big) = -\frac{1}{2}\dot{\alpha}(s)\left(f + 2x\partial_x f\right)$$

with Cauchy data $f(0,x) = \varphi_k(x)$. Make the change of variables $\sigma = \alpha(s)$ and set $g(\sigma) = f(s)$. Therefore g satisfies $\partial_\sigma g = -\frac{1}{2}(g + 2x\partial_x g)$. The equation $x = \dot{x}$, $x(0) = x_0$ admits the solution $x(\tau) = x_0\mathrm{e}^{\tau}$ and the characteristics method gives $g(\tau, x(\tau)) = \mathrm{e}^{-\frac{1}{2}\tau}\varphi_k(x_0) = \mathrm{e}^{-\frac{1}{2}\tau}\varphi_k(x(\tau)\mathrm{e}^{-\tau})$, hence

$$f(s) = \mathrm{e}^{-\frac{1}{2}\alpha(s)}\varphi_k(x\mathrm{e}^{-\alpha(s)}).$$

Corollary 2.8 *Let $k \in \mathbb{N}$, define $\omega_1 = \frac{1}{2}(\omega - 1)$ and $\mathcal{E}_{k_0} = E_0(k) = -\frac{1}{4\pi}\lambda + \frac{1}{2}k(\omega_1 - \frac{\lambda}{\pi})$. Then*

$$\begin{aligned} w_k &= e^{-is\mathcal{E}_{k_0}} U(s,0)\varphi_k \\ &= e^{-is\mathcal{E}_{k_0}} e^{i\dot{\alpha}(s)x^2/2} e^{-i(\frac{1}{2}+k)\beta(s)} e^{-\frac{1}{2}\alpha(s)} \varphi_k\left(xe^{-\alpha(s)}\right) \end{aligned} \tag{2.15}$$

is solution of the equation (2.6).

Proof: On the one hand, from Proposition 2.7 we deduce

$$\begin{aligned} w_k(s+2\pi, x) &= \mathrm{e}^{-2i\pi\mathcal{E}_{k_0}}\mathrm{e}^{-i\lambda(\frac{1}{2}+k)} w_k(s,x) = \mathrm{e}^{-ik\omega_1\pi} w_k(s,x) \\ &= (-1)^{k\omega_1} w_k(s,x) = w_k(s, \omega x). \end{aligned}$$

On the other hand, w_k satisfies (2.6) because of the definition of $U(s,0)$.

Fix $k_0 \in \mathbb{N}$ and take $v_0 = w_{k_0}$ with the previous choice of $\mathcal{E}_{k_0} = E_0(k_0)$. This choice corresponds to the k_0th level of energy for the harmonic oscillator.

Remark 2.9 *Until now we didn't use the restriction* (1.10)*, but it will crucial in the following.*

Proposition 2.10 *For all $p \geq 0$, there exist $E_p \in \mathbb{C}$ and $v_p \in \mathcal{C}^\infty([0, 2\pi], \mathcal{S}(\mathbb{R}))$ which solve (2.8).*

Remark 2.11 *As stated in Theorem 1.7, the E_j's are in fact real numbers. This will be proved in Lemma 2.14.*

Proof: We proceed by induction on $p \in \mathbb{N}$.
For $p = 0$ the result was proved in Corollary 2.8.
Let $p \geq 1$, and suppose that for all $j \leq p-1$ there exist $E_j \in \mathbb{C}$ and $v_j \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$ which solve the $(j+1)$th equation of (2.6). When $p \geq 2$, set

$$\begin{aligned}
\tilde{v}_{p-1} &= h^{\frac{1}{2}} v_1 + \cdots + h^{\frac{p-1}{2}} v_{p-1}, \\
\tilde{E}_{p-1} &= h^{\frac{1}{2}} E_1 + \cdots + h^{\frac{p-1}{2}} E_{p-1}
\end{aligned}$$

and $\tilde{v}_0 = \tilde{E}_0 = 0$. The function Q_p given by (2.8) is the coefficient of $h^{\frac{p}{2}}$ in the expansion in h of

$$\tilde{E}_{p-1}\tilde{v}_{p-1} + \frac{1}{2}\varepsilon\delta^2|v_0 + \tilde{v}_{p-1}|^2(v_0 + \tilde{v}_{p-1}) + hP(v_0 + \tilde{v}_{p-1}).$$

Now using the regularity of the v_j's and the fact that P defined by (2.5) is an operator

$$P : \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R})) \longrightarrow \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R})),$$

we obtain $Q_p \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$.
Moreover Q_p satisfies, $\forall (s,x) \in [0,2\pi] \times \mathbb{R}$

$$Q_p(s + 2\pi, x) = Q_p(s, \omega x)$$

because this property holds for the v_j's, a and for the coefficients of P.
Define $F_p(s,x) = \mathrm{e}^{-i\dot{\alpha}(s)\mathrm{e}^{2\alpha(s)}x^2/2} Q_p(s, xe^{\alpha(s)})$, then $F_p \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$ and satisfies $Q_p(s,x) = \mathrm{e}^{i\dot{\alpha}(s)x^2/2} F_p(s, x\mathrm{e}^{-\alpha(s)})$ and $F_p(s+2\pi, x) = F_p(s, \omega x)$. Let us decompose F_p on the basis $(\varphi_j)_{j\geq 0}$: there exists a unique family of smooth functions $(g_j^p(s))_{j\geq 0} \in l^2(\mathbb{N})$ so that

$$F_p(s,y) = \sum_{j\geq 0} g_j^p(s)\varphi_j(y). \tag{2.16}$$

Then

$$Q_p(s,x) = \sum_{j\geq 0} g_j^p(s)\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\varphi_j(x\mathrm{e}^{-\alpha(s)}) = \sum_{j\geq 0} h_j^p(s) w_j(s,x) \tag{2.17}$$

where according to (2.15)

$$h_j^p(s) = \mathrm{e}^{isE_0(j)}\mathrm{e}^{i(\frac{1}{2}+j)\beta(s)}\mathrm{e}^{\frac{1}{2}\alpha(s)} g_j^p(s). \tag{2.18}$$

We have

$$Q_p(s,\omega x) = \sum_{j\geq 0} h_j^p(s) w_j(s,\omega x),$$

but also

$$\begin{aligned}
Q_p(s,\omega x) = Q_p(s+2\pi, x) &= \sum_{j\geq 0} h_j^p(s+2\pi) w_j(s+2\pi, x) \\
&= \sum_{j\geq 0} h_j^p(s+2\pi) w_j(s, \omega x),
\end{aligned}$$

and from the uniqueness of the h_j^p's we deduce $h_j^p(s+2\pi) = h_j^p(s)$.

We are now looking for a solution of (2.8) of the form

$$v_p(s,x) = \sum_{j\geq 0} e_j^p(s) w_j(s,x) \tag{2.19}$$

where the e_j^p's are 2π-periodic functions. For all $j \geq 0$, by Lemma 2.8 we have

$$\Big(i\partial_s + \frac{1}{2}\partial_x^2 - \frac{1}{2}Rx^2\Big)\Big(e_j^p w_j\Big) = i\dot{e}_j^p w_j + \big(\mathcal{E}_{k_0} - \mathcal{E}_j\big) e_j^p w_j,$$

hence we have to solve the equations

$$i\dot{e}_j^p + \big(\mathcal{E}_{k_0} - \mathcal{E}_j\big) e_j^p = h_j^p + \delta_{j,k_0} E_p. \tag{2.20}$$

As $\mathcal{E}_{k_0} - \mathcal{E}_j = E_0(k_0) - E_0(j) = \frac{1}{2}(k_0 - j)(\omega_1 - \frac{\lambda}{\pi})$, the solutions of (2.20) take the form

$$e_j^p(s) = \mathrm{e}^{\frac{1}{2}i(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\Big(C_j^p - i\int_0^s h_j^p(\tau)\mathrm{e}^{-\frac{1}{2}i(k_0-j)(\omega_1-\frac{\lambda}{\pi})\tau}\mathrm{d}\tau\Big) \tag{2.21}$$

for $j \neq k_0$, and

$$e_{k_0}^p(s) = C_{k_0}^p - i\int_0^s h_{k_0}^p(\tau)\mathrm{d}\tau - iE_p s.$$

The constants $C_j^p \in \mathbb{C}$ and $E_p \in \mathbb{C}$ have to be determined such that $e_j^p(s+2\pi) = e_j^p(s)$.

• Case $j = k_0$:

$$e_{k_0}^p(s+2\pi) = -i\int_0^{2\pi} h_{k_0}^p(\tau)\mathrm{d}\tau - 2\pi i E_p + e_{k_0}^p(s),$$

thus $e_{k_0}^p$ is 2π-periodic iff

$$E_p = -\frac{1}{2\pi}\int_0^{2\pi} h_{k_0}^p \mathrm{d}\tau. \tag{2.22}$$

• Case $j \neq k_0$:
Denote by $\tilde{h}_j^p : \tau \longmapsto h_j^p(\tau)\mathrm{e}^{-i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\tau}$ and by $K = \mathrm{e}^{i(k_0-j)(\pi\omega_1-\lambda)}$. Then

$$\begin{aligned}\int_0^{s+2\pi}\tilde{h}_j^p(\tau)\mathrm{d}\tau &= \int_0^{2\pi}\tilde{h}_j^p(\tau)\mathrm{d}\tau + \int_{2\pi}^{s+2\pi}\tilde{h}_j^p(\tau)\mathrm{d}\tau \\ &= \int_0^{2\pi}\tilde{h}_j^p(\tau)\mathrm{d}\tau + K^{-1}\int_0^{s}\tilde{h}_j^p(\tau)\mathrm{d}\tau,\end{aligned}$$

and by (2.21)

$$\begin{aligned}e_j^p(s+2\pi) &= K\mathrm{e}^{-i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\Big(C_j^p - i\int_0^{s+2\pi}\tilde{h}_j^p(\tau)\mathrm{d}\tau\Big) \\ &= \mathrm{e}^{-i\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})s}\Big(KC_j^p - iK\int_0^{2\pi}\tilde{h}_j^p(\tau)\mathrm{d}\tau - i\int_0^{s}\tilde{h}_j^p(\tau)\mathrm{d}\tau\Big).\end{aligned} \tag{2.23}$$

Notice that $K \neq 1$, as $\lambda \notin \pi\mathbb{Q}$ and choose

$$C_j^p = \frac{iK}{K-1}\int_0^{2\pi} \tilde{h}_j^p(\tau)\mathrm{d}\tau,$$

then according to (2.21)and (2.23), the function e_j^p is $2\pi-$periodic.
Now, we show that the constants C_j^p are uniformly bounded in $j \geq 0$, so that the function v_p given by (2.19) is well defined. We first need the

Lemma 2.12 *Let* $(h_j^p)_{j\geq 0} \in l^2(\mathbb{N})$ *be the family of* $2\pi-$*periodic functions defined by* (2.18) *and* $h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p e^{ils}$ *its Fourier decomposition. Then for all* $n_1, n_2 \in \mathbb{N}$ *there exists* $C^p > 0$ *such that for all* $j \in \mathbb{N}$

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2} |c_{l,j}^p|^2 \leq C^p.$$

Proof: Consider the function $F_p \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$ which defines the family $(g_j^p(s))_{j\geq 0} \in l^2(\mathbb{N})$ with (2.16). Denote by $H = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$. Let $n_1, n_2 \in \mathbb{N}$ and decompose the function $\partial_s^{n_2} H^{n_1} F_p$ on the basis $(\varphi_j)_{j\geq 0}$

$$\partial_s^{n_2} H^{n_1} F_p = \sum_{j\geq 0} \tilde{g}_j^p(s)\varphi_j(s)$$

where $(\tilde{g}_j^p)_{j\geq 0}$ is a smooth family of functions in $l^2(\mathbb{N})$.
Using that $H\varphi_j = (j+\frac{1}{2})\varphi_j$ and that $F_p \in \mathcal{C}^\infty([0,2\pi], \mathcal{S}(\mathbb{R}))$, we have for all $n_1, n_2 \in \mathbb{N}$

$$\partial_s^{n_2} H^{n_1} F_p = \sum_{j\geq 0} (j+\frac{1}{2})^{n_1} (g_j^p)^{(n_2)}(s)\varphi_j(y).$$

By uniqueness of such a decomposition,

$$\Big((j+\frac{1}{2})^{n_1}(g_j^p)^{(n_2)}\Big)_{j\geq 0} = (\tilde{g}_j^p)_{j\geq 0} \in l^2(\mathbb{N})$$

Then by the definition (2.18) of h_j^p, an easy induction on $n_1, n_2 \in \mathbb{N}$ shows that $\big(j^{n_1}(h_j^p)^{(n_2)}\big)_{j\geq 0} \in l^2(\mathbb{N})$. Write the Fourier decomposition of h_j^p

$$h_j^p(s) = \sum_{n\in\mathbb{Z}} c_{l,j}^p \mathrm{e}^{ils}$$

and by Parseval

$$\sum_{j\geq 0}\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2}(s)|c_{l,j}^p|^2 = \sum_{j\geq 0} j^{2n_1}\int_0^{2\pi} |(h_j^p)^{(n_2)}(s)|^2 \mathrm{d}s \leq C^p.$$

In particular, for all $j \in \mathbb{N}$

$$\sum_{l\in\mathbb{Z}} j^{2n_1} l^{2n_2} |c_{l,j}^p|^2 \leq C^p,$$

hence the result.

End of the proof of Proposition 2.10: Using the Fourier decomposition of h_j we obtain

$$\begin{aligned} C_j^p &= \frac{iK}{K-1}\int_0^{2\pi} \tilde{h}_j^p(\tau)\mathrm{d}\tau \\ &= \frac{iK}{K-1}\sum_{l\in\mathbb{Z}} c_{l,j}^p \int_0^{2\pi} \mathrm{e}^{i(l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi}))\tau} \\ &= -i\sum_{l\in\mathbb{Z}} \frac{c_{l,j}^p}{l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})}. \end{aligned} \tag{2.24}$$

With Assumption 1.5 we have

$$\begin{aligned} \big|l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\big| &= \frac{1}{2}|(2l-(k_0-j)\omega_1)+(k_0-j)\frac{\lambda}{\pi}| \\ &\geq \frac{1}{2}\frac{\mu}{|(2l-(k_0-j)\omega_1, k_0-j)|^\tau}, \end{aligned}$$

and for $j\geq k_0$, $|2l-(k_0-j)\omega_1|+|k_0-j|\leq 2(|l|+|j|)$, then

$$\big|l-\frac{1}{2}(k_0-j)(\omega_1-\frac{\lambda}{\pi})\big| \geq \frac{\mu'}{(|l|+|j|)^\tau}. \tag{2.25}$$

Hence, from (2.24) and (2.25) we deduce

$$|C_j^p| \lesssim \sum_{l\in\mathbb{Z}} |c_{l,j}^p|(|j|+|l|)^\tau \lesssim \sum_{l\in\mathbb{Z}} |c_{l,j}^p|(|j|^\tau+|l|^\tau). \tag{2.26}$$

By Cauchy-Schwarz and Lemma 2.12, from (2.26) we obtain

$$\begin{aligned} |C_j| &\lesssim \sum_{l\in\mathbb{Z}} \frac{1+|l|}{1+|l|}|c_{l,j}^p|(|j|^\tau+|l|^\tau) \\ &\lesssim \big(\sum_{l\in\mathbb{Z}} \frac{1}{(1+|l|)^2}\big)^{\frac{1}{2}}\big(\sum_{l\in\mathbb{Z}} |c_{l,j}^p|^2(1+|l|)^2(|j|^{2\tau}+|l|^{2\tau})\big)^{\frac{1}{2}} \\ &\leq C^p. \end{aligned} \tag{2.27}$$

Set

$$v_p(s,x) = \sum_{j\geq 0} e_j^p(s) w_j(s,x).$$

For all $j\in\mathbb{N}$, $s\longmapsto e_j^p(s)w_j(s,x)$ is continuous and there exists $c>0$ such that for all $j>k_0$, and for all $s\in[0,2\pi]$

$$|e_j^p(s)w_j(s,x)| \lesssim |g_j^p(s)||\varphi_j(cx)|$$

and this shows that $v_p\in C\big([0,2\pi],L^2(\mathbb{R})\big)$. Now using Proposition 2.6 we conclude, by uniqueness of such a solution, that $v_p\in\mathcal{C}^\infty([0,2\pi],\mathcal{S}(\mathbb{R}))$.

2.2 The nonlinear analysis and proof of Proposition 2.2

LEMMA 2.13 *The constant E_1 given by Proposition 2.10 writes $E_1=-\varepsilon\delta^2 C_0$ where $C_0>0$ is independent of ε and δ.*

Proof: According to formula (2.22), we only have to compute the term h_{k_0} in the expansion (2.17).
Write the expansion of $|\varphi_{k_0}|^2\varphi_{k_0}$ on the basis $(\varphi_j)_{j\geq 0}$:

$$|\varphi_{k_0}|^2\varphi_{k_0} = \sum_{j\geq 0} p_j\varphi_j, \tag{2.28}$$

with $p_j \in \mathbb{R}$ and $p_j = 0$ for $j - k_0 = 1$ *mod* 2 as $\varphi_k(-x) = (-1)^k\varphi_k(x)$. Then

$$\begin{aligned}
|v_0|^2 v_0 &= \mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{3}{2}\alpha(s)}|\varphi_{k_0}|^2\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right)\\
&= \sum_{j\geq 0} p_j\mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{-\frac{3}{2}\alpha(s)}\varphi_j\left(x\mathrm{e}^{-\alpha(s)}\right)\\
&= \sum_{j\geq 0} f_j(s)w_j(s,x)
\end{aligned}$$

where

$$\begin{aligned}
f_j(s) &= p_j\mathrm{e}^{-is(E_0(k_0)-E_0(j))}\mathrm{e}^{-i(k_0-j)\beta(s)}\mathrm{e}^{-\alpha(s)}\\
&= p_j\mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)}\mathrm{e}^{-\alpha(s)}.
\end{aligned}$$

Therefore $f_{k_0}(s) = p_{k_0}\mathrm{e}^{-\alpha(s)}$ with, using (2.28), $p_{k_0} = \int_{\mathbb{R}}|\phi_{k_0}|^4 > 0$.
In the same manner we write

$$x^3\varphi_{k_0}(x) = \sum_{j\geq 0} q_j\varphi_j(x)$$

with $q_j = 0$ when $j - k_0 = 0$ *mod* 2 and

$$\begin{aligned}
R_3x^3v_0 &= R_3\mathrm{e}^{-isE_0(k_0)}\mathrm{e}^{i\dot{\alpha}(s)x^2/2}\mathrm{e}^{-i(\frac{1}{2}+k_0)\beta(s)}\mathrm{e}^{\frac{5}{2}\alpha(s)}(x\mathrm{e}^{-\alpha(s)})^3\varphi_{k_0}\left(x\mathrm{e}^{-\alpha(s)}\right)\\
&= \sum_{j\geq 0} g_j(s)w_j(s,x)
\end{aligned}$$

where

$$\begin{aligned}
g_j(s) &= q_jR_3(s)\mathrm{e}^{-is(E_0(k_0)-E_0(j))}\mathrm{e}^{-i(k_0-j)\beta(s)}\mathrm{e}^{3\alpha(s)}\\
&= q_jR_3(s)\mathrm{e}^{-i(k_0-j)(\theta(s)+\frac{s}{2}\omega_1)}\mathrm{e}^{3\alpha(s)}.
\end{aligned}$$

Then $g_{k_0} = 0$ and $h_{k_0} = \frac{1}{2}\varepsilon\delta^2 f_{k_0} = \frac{1}{2}\varepsilon\delta^2 p_{k_0}\mathrm{e}^{-\alpha(s)}$.
Finally, from (2.22) we deduce

$$E_1 = -\frac{1}{4\pi}\varepsilon\delta^2 p_{k_0}\int_0^{2\pi}\mathrm{e}^{-\alpha(\tau)}\mathrm{d}\tau = -\varepsilon\delta^2 C_0$$

where $C_0 > 0$ as $p_{k_0} > 0$.

Now we prove Proposition 2.2

Proof of Proposition 2.2: To begin with, as χ is an even function and by construction of the v_j's, $u_p(s+2\pi,x) = u_p(s,\omega x)$.
Set $v(s,x) = \left(v_0 + h^{\frac{1}{2}}v_1 + \cdots + h^{\frac{p}{2}}v_p\right)(s,x)$, then plugging in (2.4) we obtain

that the coefficient of $h^{\frac{j}{2}}$ cancel for $0 \leq j \leq p$, this leads to an error term $h^{\frac{p+1}{2}} F_h(s,x)$ with $F_h \in \mathcal{C}^\infty\left([0,2\pi]\times]-\frac{r_0}{\sqrt{h}},\frac{r_0}{\sqrt{h}}[,\mathbb{R}\right)$. Note that the function F_h isn't defined for $x\in\mathbb{R}$, as it depends on a which is only defined for $r\in]-r_0,r_0[$. Set $\tilde{u}_p = \delta e^{\frac{i}{h}s} v(s,\frac{r}{\sqrt{h}})$, then coming back to equation (2.1), there exists $G_h \in \mathcal{C}^\infty\left([0,2\pi]\times]-\frac{r_0}{\sqrt{h}},\frac{r_0}{\sqrt{h}}[,\mathbb{R}\right)$ so that

$$-\Delta\tilde{u}_p - \lambda_p\tilde{u}_p + \varepsilon|\tilde{u}_p|^2\tilde{u}_p = \delta h^{\frac{p-1}{2}} h^{-\frac{1}{4}} G_h.$$

(Here we loose a power of h as we had multiplied by h to obtain (2.4).)
Now

$$\begin{aligned} -\Delta u_p = \lambda_p u_p + \varepsilon|u_p|^2 u_p &= \chi(-\Delta\tilde{u}_p - \lambda_p\tilde{u}_p) + \varepsilon\chi^3|\tilde{u}_p|^2\tilde{u}_p \\ &\quad -\chi'(2\tilde{u}_p + \frac{\partial_r a}{a}\tilde{u}_p) - \chi''\tilde{u}_p \\ &= h^{\frac{p-1}{2}} h^{-\frac{1}{4}}\chi G_h + \varepsilon\chi(\chi^2-1)|\tilde{u}_p|^2\tilde{u}_p \\ &\quad -\chi'(2\tilde{u}_p + \frac{\partial_r a}{a}\tilde{u}_p) - \chi''\tilde{u}_p \qquad (2.29) \\ &:= h^{\frac{p-1}{2}} g_p(h). \end{aligned}$$

First, observe that

$$\begin{aligned} h^{-\frac{1}{2}}\int_{[0,2\pi]\times\mathbb{R}} \chi^2(r)\left|G_h(s,\frac{r}{\sqrt{h}})\right|^2 \mathrm{d}s\mathrm{d}r &= \int_{[0,2\pi]\times\mathbb{R}} \chi^2(\sqrt{h}x)\left|G_h(s,x)\right|^2 \mathrm{d}s\mathrm{d}x \\ &\sim 1. \qquad (2.30) \end{aligned}$$

Then, as v is rapidly decreasing in x, $\tilde{u}_p$ is localized near $r=0$ in an interval of length $\sim\sqrt{h}$. But each of the terms $\chi(\chi^2-1)$, χ' and χ'' vanish when $|r|\leq r_0/2$. Thus, for all $N\in\mathbb{N}$ there exists C_N such that

$$\begin{aligned} \|\chi(\chi^2-1)|\tilde{u}_p|^2\tilde{u}_p\|_{L^2(M)} &\leq C_N h^N, \\ \|\chi'(2\tilde{u}_p + \frac{\partial_r a}{a}\tilde{u}_p)\|_{L^2(M)} &\leq C_N h^N, \\ \|\chi''\tilde{u}_p\|_{L^2(M)} &\leq C_N h^N. \end{aligned}$$

These estimates together with (2.30) yield

$$\|g_p(h)\|_{L^2(M)} \lesssim \delta.$$

Derivating a term of the right hand side of (2.29) costs at most h^{-1} (when you derivate $\exp i\frac{s}{h}$ is variable s). Hence, for all $n\in\mathbb{N}$

$$\|g_p(h)\|_{L^2(M)} \lesssim \delta h^{-n}.$$

LEMMA 2.14 *Let $p\geq 1$ and E_p given by Proposition (2.10). Then $E_p\in\mathbb{R}$.*

Proof: We already know that $E_0, E_1\in\mathbb{R}$. Let $p\geq 3$. Multiply (2.11) by $\overline{u}_p$, integrate on M and take the imaginary part

$$0 = \mathrm{Im}\lambda_p\|u_p\|_{L^2}^2 + h^{\frac{p-1}{2}}\mathrm{Im}\int g_p(h)\overline{u}_p.$$

As $\|u_p\|_{L^2} \sim 1$ and $\|g_p\|_{L^2} \lesssim 1$, we obtain the estimate

$$|\mathrm{Im}\lambda_p| \lesssim h^{\frac{p-1}{2}} \|g_p\|_{L^2} \|u_p\|_{L^2} \lesssim h^{\frac{p-1}{2}}$$

and as

$$\mathrm{Im}\lambda_p = -2(\mathrm{Im}E_2 + h^{\frac{1}{2}} \mathrm{Im}E_2 + \cdots + h^{\frac{p-1}{2}} \mathrm{Im}E_p)$$

it follows that for all $0 \leq j \leq p-1$, $\mathrm{Im}E_j = 0$, i.e. $E_j \in \mathbb{R}$.

3 The instability for the nonlinear Schrödinger equation

3.1 The error estimate

Proposition 3.1 *Let $\alpha > 0$ and let $v \in H^2(M)$ be such that*

$$\|v\|_{L^2} \lesssim 1, \quad \|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}, \quad \|\Delta v\|_{L^\infty} \lesssim h^{-\frac{9}{4}+\sigma},$$

and suppose that v satisfies

$$i\partial_t v + \Delta v = \varepsilon |v|^2 v + h^\alpha R(h)$$

with for all $\beta \in [0,2]$, $\|R(h)\|_{H^\beta} \lesssim h^{-\beta}$. Let u be solution of

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon |u|^2 u \\ u(0,x) = v(0,x). \end{cases}$$

Then, if $\alpha > \frac{1}{4} + 3\sigma$ we have

$$\|(u-v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad \textit{when} \quad h \longrightarrow 0,$$

where $t_h \sim h^{\frac{1}{2}-2\sigma} \log(\frac{1}{h})$.

Proof: Define $w = u - v$ and

$$E(t) = \|w\|_{L^2}^2 + \|h^2 \Delta w\|_{L^2}^2.$$

We have $E(0) = 0$ and the following estimates:

$$\|w\|_{L^2} \leq E^{\frac{1}{2}}, \quad \|\Delta w\|_{L^2} \leq h^{-2} E^{\frac{1}{2}}, \quad \|\nabla w\|_{L^2} \leq h^{-1} E^{\frac{1}{2}}. \tag{3.1}$$

The function w satisfies the equation

$$i\partial_t w + \Delta w = \varepsilon(|w+v|^2(w+v) - |v|^2 v) + R(h). \tag{3.2}$$

The energy method gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &= \mathrm{Im} \int \overline{w} \left(\varepsilon(|w+v|^2(w+v) - |v|^2 v) + R(h) \right) \\ &\lesssim h^\alpha \|w\|_{L^2} + \|w\|_{L^4}^4 + \|w\|_{L^2}^2 \|v\|_{L^\infty}^2. \end{aligned}$$

The Gagliardo-Nirenberg inequality gives

$$\|w\|_{L^4}^4 \lesssim \|w\|_{L^2}^2 \|\nabla w\|_{L^2}^2 \lesssim h^{-2} E^2,$$

and as $\|v\|_{L^\infty} \lesssim h^{-\frac{1}{4}+\sigma}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 \lesssim h^\alpha E^{\frac{1}{2}} + h^{-\frac{1}{2}+2\sigma} E + h^{-2} E^2. \tag{3.3}$$

Now, apply Δ to (3.2)

$$i\partial_t \Delta w + \Delta^2 w = \varepsilon \Delta A + \Delta R(h) \tag{3.4}$$

with

$$\begin{aligned} A &= |w+v|^2(w+v) - |v|^2 v \\ &= 2w|v|^2 + \overline{w} v^2 + w^2 \overline{v} + 2|w|^2 v + |w|^2 w, \end{aligned}$$

then

$$\begin{aligned} |\Delta(A)| \lesssim\ & |v|^2|\Delta w| + |v||\nabla v||\nabla w| + |\nabla v|^2|w| + |v||\Delta v||w| \\ & + |\Delta v||w|^2 + |w|^2|\Delta w| + |w||\nabla w|^2 \end{aligned}$$

hence

$$\begin{aligned} \|\Delta(A)\|_{L^2} \lesssim\ & \|v\|_{L^\infty}^2 \|\Delta w\|_{L^2} + \|v\|_{L^\infty}\|\nabla v\|_{L^\infty}\|\nabla w\|_{L^2} + \|\nabla v\|_{L^\infty}^2 \|w\|_{L^2} \\ & + \|v\|_{L^\infty}\|\Delta v\|_{L^\infty}\|w\|_{L^2} + \|\Delta v\|_{L^\infty}\|w\|_{L^4}^2 \\ & + \|w\|_{L^\infty}^2 \|\Delta w\|_{L^2} + \|w\|_{L^2}\|\nabla w\|_{L^4}^2. \end{aligned} \tag{3.5}$$

The following inequality holds in dimension 2

$$\|w\|_{L^\infty} \lesssim \|w\|_{L^2}^{\frac{1}{2}} \|\Delta w\|_{L^2}^{\frac{1}{2}} \lesssim h^{-1} E^{\frac{1}{2}},$$

and with (3.1) and (3.5) we deduce

$$\|\Delta(A)\|_{L^2} \lesssim h^{-\frac{5}{2}+2\sigma} E^{\frac{1}{2}} + h^{-\frac{13}{4}+\sigma} E + h^{-4} E^{\frac{3}{2}}.$$

But

$$h^{-\frac{13}{4}+\sigma} E = h^{-\frac{5}{4}+\sigma} E^{\frac{1}{4}} h^{-2} E^{\frac{3}{4}} \lesssim h^{-\frac{5}{2}+2\sigma} E^{\frac{1}{2}} + h^{-4} E^{\frac{3}{2}},$$

and we obtain

$$\|\Delta(A)\|_{L^2} \lesssim h^{-\frac{5}{2}+2\sigma} E^{\frac{1}{2}} + h^{-4} E^{\frac{3}{2}}. \tag{3.6}$$

Now, using (3.6) and $\|\Delta(R(h))\|_{L^2} \lesssim h^{\alpha-2}$, the energy method and the Cauchy-Schwarz inequality gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta w\|_{L^2}^2 &= \operatorname{Im}\int \Delta\overline{w}\,(A + R(h)) \\ &\lesssim h^{-2} E^{\frac{1}{2}} (h^{\alpha-2} + h^{-\frac{5}{2}+2\sigma} E^{\frac{1}{2}} + h^{-4} E^{\frac{3}{2}}), \end{aligned} \tag{3.7}$$

therefore from (3.3) and (3.7) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E \lesssim h^\alpha E^{\frac{1}{2}} + h^{-\frac{1}{2}+2\sigma} E + h^{-2} E^2.$$

Interpolation gives

$$\|w\|_{H^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{\dot{H}^\sigma} \lesssim \|w\|_{L^2} + \|w\|_{L^2}^{1-\frac{\sigma}{2}} \|\Delta w\|_{L^2}^{\frac{\sigma}{2}} \lesssim h^{-\sigma} E^{\frac{1}{2}} := F$$

and F satisfies $F(0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} F \lesssim h^{-\sigma+\alpha} + h^{-\frac{1}{2}+2\sigma} F + h^{-2+2\sigma} F^3. \tag{3.8}$$

As long as $h^{-2+2\sigma}F^3 \lesssim h^{-\frac{1}{2}+2\sigma}F$, i.e. $F \lesssim h^{\frac{3}{4}}$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} F \lesssim h^{-\sigma+\alpha} + h^{-\frac{1}{2}+2\sigma} F$$

and the Gronwall inequality yields

$$F \lesssim h^{\alpha+\frac{1}{2}-3\sigma} \mathrm{e}^{C+h^{-\frac{1}{2}+2\sigma} t}.$$

The non linear term in (3.8) can be removed with the continuity argument for times such that

$$h^{\alpha+\frac{1}{2}-3\sigma} \mathrm{e}^{C+h^{-\frac{1}{2}+2\sigma} t} \lesssim h^{\frac{3}{4}+\eta}$$

with $\eta > 0$ i.e. for $t \lesssim (\alpha - \frac{1}{4} - 3\sigma - \eta) h^{-\frac{1}{2}+2\sigma} \log \frac{1}{h}$, which is possible with η small enough as we assume $\alpha > \frac{1}{4} + 3\sigma$.

Corollary 3.2 *Let $\kappa > 0$, $0 < \sigma < \frac{1}{4}$ and set $\delta = \kappa h^\sigma$. Denote by $v = e^{-i\lambda_3 t} u_3$ where u_3 and λ_3 are defined by (2.9) and (2.10) respectively.*
Let u be solution of

$$\begin{cases} i\partial_t u + \Delta u = \varepsilon |u|^2 u \\ u(0,x) = v(0,x). \end{cases}$$

Then $\|v\|_{H^\sigma} \sim 1$ and

$$\|(u - v)(t_h)\|_{H^\sigma} \longrightarrow 0 \quad \textit{when} \quad h \longrightarrow 0,$$

where $t_h \sim h^{\frac{1}{2}-2\sigma} \log(\frac{1}{h})$.

Proof: The result directly follows from Propositions (2.2) and (3.1), as for all $0 < \sigma < \frac{1}{4}$, we have $\sigma + 1 > \frac{1}{4} + 3\sigma$.

3.2 The instability argument

Let $\kappa, \kappa_h > 0$ and consider $v = v^1$ defined in Corollary 3.2 associated with κ and v^2 associated with κ_h. Let u be a solution of

$$\begin{cases} i\partial_t u^j + \Delta u^j = \varepsilon |u^j|^2 u^j \\ u^j(0,x) = v^j(0,x) \end{cases}$$

and $t_h \sim h^{\frac{1}{2}-2\sigma} \log(\frac{1}{h})$. Then

$$\begin{aligned} \|(u^2 - u^1)(t_h)\|_{H^\sigma} \quad &\geq \quad \|(v^2 - v^1)(t_h)\|_{H^\sigma} - \|(u^2 - v^2)(t_h)\|_{H^\sigma} \\ &\quad -\|(u^1 - v^1)(t_h)\|_{H^\sigma}. \end{aligned} \tag{3.9}$$

From Corollary 3.2 we deduce that for $j = 1, 2$

$$\|(u^j - v^j)(t_h)\|_{H^\sigma} \longrightarrow 0. \tag{3.10}$$

Observe that

$$\|(v^2 - v^1)(t_h)\|_{H^\sigma} \sim \left|\mathrm{e}^{-i\lambda_3^2 t_h} - \mathrm{e}^{-i\lambda_3^1 t_h}\right| = \left|\mathrm{e}^{i(\lambda_3^2 - \lambda_3^1)t_h} - 1\right|,$$

from Proposition 2.13 we have

$$(\lambda_3^2 - \lambda_3^1)t_h \sim h^{2\sigma-1}(\kappa - \kappa_h)t_h \sim (\kappa - \kappa_h)\log\frac{1}{h}.$$

It is possible to choose κ_h such that $\kappa_h \longrightarrow \kappa$ and $(\kappa - \kappa_h)\log\frac{1}{h} \longrightarrow \infty$. Then using (3.9) and (3.10)

$$\limsup_{h\longrightarrow 0} \|(u^2 - u^1)(t_h)\|_{H^\sigma} \geq \limsup_{h\longrightarrow 0} \|(v^2 - v^1)(t_h)\|_{H^\sigma} \geq 2$$

even though

$$\|(u^2 - u^1)(0)\|_{H^\sigma} = \|(v^2 - v^1)(0)\|_{H^\sigma} \sim |\kappa - \kappa_h|$$

which tends to 0 with h. According to Definition 1.1, we have proved Proposition 1.3.

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